THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1520G/H University Mathematics 2014-2015
Suggested Solution to Assignment 5

Exercise 1:

Use integration by part:

$$\begin{aligned}
I_n - I_{n-1} &= -\int_0^1 x^3(1-x^3)^{n-1}dx \\
&= -\frac{1}{3}\int_0^1 x\cdot 3x^2(1-x^3)^{n-1}dx \\
&= \frac{1}{3}\int_0^1 x(1-x^3)^{n-1}d(1-x^3) \\
&= \frac{1}{3n}\int_0^1 xd(1-x^3)^n \\
&= \frac{1}{3n}x(1-x^3)^n|_0^1 - \frac{1}{3n}\int_0^1 (1-x^3)^n dx \\
&= -\frac{1}{3n}I_n
\end{aligned}$$

Which is $(3n+1)I_n = 3nI_{n-1}$.

Exercise 2:

By the definition of Riemann integration:

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{2\pi}{n}+\sin\frac{4\pi}{n}+\cdots+\frac{2n\pi}{n}\right) &= \frac{1}{2\pi}\lim_{n\to\infty}\frac{2\pi}{n}\left(\sin\frac{2\pi}{n}+\sin\frac{4\pi}{n}+\cdots+\frac{2n\pi}{n}\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi}\sin x dx \\
&= 0
\end{aligned}$$

Exercise 3:

Use substitutions $t = \frac{x}{2}$, $s = \tan t$, $y = s + \frac{4}{5}$:

$$\begin{aligned}
\int \frac{dx}{5+4\sin x} &= \int \frac{dx}{5+8\sin\frac{x}{2}\cos\frac{x}{2}} \\
&= \int \frac{dx}{5\sin^2\frac{x}{2}+8\sin\frac{x}{2}\cos\frac{x}{2}+5\cos^2\frac{x}{2}} \\
&= \int \frac{2dt}{5\sin^2 t+8\sin t\cos t+5\cos^2 t} \\
&= \int \frac{2\sec^2 t dt}{5\tan^2 t+8\tan t+5} \\
&= \int \frac{2d\tan t}{5\tan^2 t+8\tan t+5} \\
&= \int \frac{2ds}{5s^2+8s+5} \\
&= \int \frac{2ds}{5s^2+8s+5} \\
&= \frac{2}{5}\int \frac{d(s+\frac{4}{5})}{(s+\frac{4}{5})^2+\frac{9}{25}} \\
&= \frac{2}{5}\int \frac{dy}{y^2+\frac{9}{25}} \\
&= \frac{2}{3}\tan^{-1}\frac{5y}{3}+C
\end{aligned}$$

Then we find: $\displaystyle\int \frac{dx}{5+4\sin x} = \frac{2}{3}\tan^{-1}\frac{5}{3}\left(\tan\frac{x}{2}+\frac{4}{5}\right)+C$

Exercise 4:

(a)

$$\begin{aligned}
\int_0^1 \frac{u^4(1-u)^4}{1+u^2}du &= \int_0^1 \frac{(1-u^4)-(1-u^4)(1-u)^4}{1+u^2}du \\
&= \int_0^1 \frac{(1-u)^4}{1+u^2}du - \int_0^1 (1-u^4)(1-u)^4 du \\
&= \int_0^1 \frac{(1-2u+u^2)^2}{1+u^2}du - \int_0^1 (1+u)(1-u)^5 du \\
&= \int_0^1 \frac{(1+u^2)^2-4u(1+u^2)+4u^2}{1+u^2}du - \int_0^1 (1+1-x)x^5 dx \\
&= \int_0^1 (1+u^2)-4u+4-\frac{4}{1+u^2}du - 2\int_0^1 x^5 dx + \int_0^1 x^6 dx \\
&= 1+\frac{1}{3}-2+4-\pi-\frac{1}{3}+\frac{1}{7} \\
&= \frac{22}{7}-\pi
\end{aligned}$$

(b)

$$\frac{1}{2}u^4(1-u)^4 \le \frac{u^4(1-u)^4}{1+u^2} \le u^4(1-u)^4$$

$$\int_0^1 \frac{1}{2}u^4(1-u)^4 du \le \int_0^1 \frac{u^4(1-u)^4}{1+u^2}du \le \int_0^1 u^4(1-u)^4 du$$

$$
\begin{aligned}
\int_0^1 u^4(1-u)^4 du &= \int_0^1 u^4 - 4u^5 + 6u^6 - 4u^7 + u^8 du \\
&= \frac{1}{5} - \frac{2}{3} + \frac{6}{7} - \frac{1}{2} + \frac{1}{9} \\
&= \frac{1}{630}
\end{aligned}
$$

Finally we get:

$$
\frac{1}{1260} \le \int_0^1 \frac{u^4(1-u)^4}{1+u^2} du \le \frac{1}{630}
$$

Then:

$$
\frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630} \Longrightarrow \frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}
$$

Exercise 5:

(a).$f^{'}(x) = [(1-x)^n - n(1-x)^{n-1}]e^x = (1-x)^{n-1}(1-n-x)e^x \le 0$ for $0 \le x \le 1$ and $n \ge 1$. So $f(x)$ is decreasing.

(b).Use integration by part n times:

$$
\begin{aligned}
\int_0^1 (1-x)^n e^x dx &= \int_0^1 (1-x)^n de^x \\
&= (1-x)^n e^x|_0^1 + n\int_0^1 (1-x)^{n-1} e^x dx \\
&= -1 + n\int_0^1 (1-x)^{n-1} e^x dx \\
&\dots \\
&= -1 - n - n(n-1) - \dots - n! + n!\int_0^1 e^x dx \\
&= -1 - n - n(n-1) - \dots - n! + n!(e-1)
\end{aligned}
$$

$$
\frac{1}{n!}\int_0^1 f(x)dx = e - \sum_{r=1}^{n} \frac{1}{r!}
$$

(c).By(a)we know $0 \le f(x) \le 1$, so $0 \le \int_0^1 f(x)dx \le 1$. Use (b) we find:

$$
0 \le e - \sum_{r=1}^{n} \frac{1}{r!} \le \frac{1}{n!}
$$

Let $n \to \infty$ we get $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{r!} = e$.